EXPLICIT FORMULAE AND DISCREPANCY ESTIMATES FOR a-POINTS OF THE RIEMANN ZETA-FUNCTION

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Abstract. For a fixed $a \neq 0$, an a-point of the Riemann zeta-function is a complex number $\rho_a = \beta_a + i\gamma_a$ such that $\zeta(\rho_a) = a$. Recently J. Steuding estimated the sum

$$\sum_{\substack{0<\gamma_a\leq T\\ \beta_a>0}} x^{\rho_a}$$

for a fixed x as $T \to \infty$, and used this to prove that the ordinates γ_a are uniformly distributed modulo 1. We provide uniform estimates for this sum when $x > 0$ and $\neq 1$, and $T > 1$. Using this, we bound the discrepancy of the sequence $\lambda\gamma_a$ when $\lambda \neq 0$. We also find explicit representations and bounds for the Dirichlet coefficients of the series $1/(\zeta(s)-a)$ and upper bounds for the abscissa of absolute convergence of this series.

1. Introduction and Results

Let $\zeta(s)$ denote the Riemann zeta-function, where $s = \sigma + it$ is a complex variable. As is usual, we shall denote zeros of the zeta-function by $\rho = \beta + i\gamma$. If a is a nonzero complex number, an *a-point* of $\zeta(s)$ is a number $\rho_a = \beta_a + i\gamma_a$ such that $\zeta(\rho_a) = a$. That is, it is a zero of $F(s) = \zeta(s) - a$. For basic results about a-points we refer the reader to [10], [12], and [15]. In particular, it is known that there exists a number $n_0(a)$ such that for each $n \geq n_0(a)$ there is an a-point very close to $s = -2n$, and there are at most finitely many other a-points in $\sigma \leq 0$. We call these the *trivial* a-points, and the remaining a-points *nontrivial*. Since a Dirichlet series that is not identically zero has a right half-plane free of zeros, the nontrivial a-points lie in a strip $0 < \sigma < A$, where A depends on a. It was proved in the paper of Bohr, Landau, and Littlewood [2] that the number of these with $0 < \gamma_a \leq T$ is

$$N_a(T) = \sum_{\substack{0<\gamma_a\leq T\\ \beta_a>0}} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O_a(\log T) \tag{1.1}$$

provided that $a \neq 1$; if $a = 1$ there is an additional term $-\log 2(T/2\pi)$ on the right-hand side. The corresponding formula for the number of nontrivial zeros of the zeta-function is

$$N(T) = \sum_{0<\gamma\leq T} 1 = \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi} + O(\log T).$$

It was also proved in [2] that if the Riemann hypothesis is true, the a-points cluster about the line $\sigma = 1/2$. Much later Levinson [10] showed that this holds unconditionally. A similar clustering result was proved for the zeros of the zeta-function by Bohr and Landau [1]. Despite these similarities, there is a striking difference between the distribution of a-points and zeros: for each fixed σ with $1/2 < \sigma \leq 1$ the number of a-points with $\beta_a > \sigma$ and $0 < \gamma_a \leq T$ is $\gg T$, whereas $\zeta(s)$ has only $o(T)$ zeros in this region (Titchmarsh [15]).

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In 1911 Landau [8] proved the remarkable formula

$$\sum_{0<\gamma\le T} x^{\rho} \;=\; -\Lambda(x)\frac{T}{2\pi} + O(\log T) \qquad (T\to\infty),$$

where $x>1$ is fixed. Here $\Lambda(x)$ is von Mangoldt's function defined as $\Lambda(n)=\log p$ if $n=p^k$ for some natural number k, and $\Lambda(x)=0$ for all other real x. A formula for $0<x<1$ follows on replacing x by $1/x$, multiplying the resulting sum by x, and observing that $1-\rho$ runs through the nontrivial zeros as ρ does. The two x-ranges may be combined and stated as

$$\sum_{0<\gamma\le T} x^{\rho} \;=\; -\Big(\Lambda(x)+x\Lambda\Big(\frac{1}{x}\Big)\Big)\frac{T}{2\pi} + O(\log T) \qquad (T\to\infty), \tag{1.2}$$

for any fixed positive $x\ne 1$. Recently, Steuding [13, Theorem 6] proved an analogous formula for a-points, namely,

$$\sum_{0<\gamma_a\le T} x^{\rho_a} \;=\; -\Big(\Lambda_a(x)+x\Lambda\Big(\frac{1}{x}\Big)\Big)\frac{T}{2\pi} + O(T^{1/2+\varepsilon}), \tag{1.3}$$

where $x\ne 1$ is fixed and positive and $\varepsilon>0$ is arbitrarily small. When $a\ne 1$, $\Lambda_a(n)$ is defined for integers $n\ge 2$ by means of the Dirichlet series

$$-\frac{\zeta'(s)}{\zeta(s)-a} \;=\; \sum_{n=2}^{\infty}\frac{\Lambda_a(n)}{n^s}. \tag{1.4}$$

For other real x, $\Lambda_a(x)=0$. When $a=1$, Λ_a is defined for numbers $m2^r$ with m an odd positive integer and r any integer by means of the generalized Dirichlet series

$$-\frac{\zeta'(s)}{\zeta(s)-1} \;=\; \sum_{\substack{m=1\\ \text{odd}}}^{\infty}\sum_{r=-\infty}^{\infty}\frac{\Lambda_1(2^r m)}{(2^r m)^s}.$$

Here too $\Lambda_1(x)=0$ for other real x.

The implied constants in (1.2) and (1.3) are highly dependent on x. For example, in the case of (1.2), Gonek [6], [7] proved that when $x,T>1$

$$\begin{aligned}\sum_{0<\gamma\le T} x^{\rho} \;&=-\frac{T}{2\pi}\Lambda(x)+O(x\log(2xT)\log\log(3x))+O\Big(\log x\;\min\Big(T,\frac{x}{\langle x\rangle}\Big)\Big)\\ &\quad+O\Big(\log(2T)\;\min\Big(T,\frac{1}{\log x}\Big)\Big),\end{aligned}$$

where $\langle x\rangle$ denotes the distance from x to the nearest prime power other than x itself, and the implied constants in the O-terms are absolute. An immediate corollary of this is that for $x,T>1$ we have

$$\begin{aligned}\sum_{0<\gamma\le T} x^{-\rho} &=-\frac{T}{2\pi x}\Lambda(x)+O(\log(2xT)\log\log(3x))+O\Big(\log x\;\min\Big(\frac{T}{x},\frac{1}{\langle x\rangle}\Big)\Big)\\ &\quad+O\Big(\log(2T)\;\min\Big(\frac{T}{x},\frac{1}{x\log x}\Big)\Big).\end{aligned}$$

Our first aim here is to prove analogues of these formulae for (1.3).

In stating our results it will be convenient to write

$$\beta_a^* = \sup_{\rho_a}\beta_a \tag{1.5}$$

and

$$B=\beta_a^*+\varepsilon,$$

where $\varepsilon>0$ is arbitrary. Thus, the value of B may be different at different occurrences. As was mentioned above, there is a number A such that all $\beta_a<A$, so β_a^* is finite. Furthermore, (see

Theorem 11.6 (C) of [15]) we know that for every $\delta > 0$ the equation $\zeta(s) = a$ has solutions in the strip $1 < \sigma < 1 + \delta$. Thus $\beta_a^* > 1$.

Theorem 1.1. *Suppose $a \neq 0, 1$ is a fixed complex number and let $x, T > 1$. Then*

$$\sum_{\substack{0<\gamma_a\leq T\\ \beta_a>0}} x^{\rho_a} \;=\; -\frac{T}{2\pi}\Lambda_a(x) \;+\; O\Big(x^B\Big(1 \;+\; \min\Big\{T, \frac{x}{\langle x\rangle}\Big\}\Big)\Big) \;+\; O\Big(x^{B+1}\log T\Big(1+\frac{1}{\log x}\Big)\Big)$$
$$+\; O\Big(\frac{\log T}{x^2}\Big(1+\min\Big\{T, \frac{1}{\log x}\Big\}\Big)\Big).$$

The implied constants depend only on a and the value of ε in the definition of B.

To estimate $\sum_{\substack{0<\gamma_a\leq T\\ \beta_a>0}} x^{\rho_a}$ when $0 < x < 1$, we consider $\sum_{\substack{0<\gamma_a\leq T\\ \beta_a>0}} x^{-\rho_a}$ with $x > 1$. In this case we do not need to exclude $a = 1$.

Theorem 1.2. *Let $a \neq 0$ and $0 < \theta < 1$ be fixed. If $T > 1$ and $1 < x \leq T^\theta$, then*

$$\sum_{\substack{0<\gamma_a\leq T\\ \beta_a>0}} x^{-\rho_a} \;=\; -\frac{T}{2\pi x}\Lambda(x) \;+\; O\Big(\frac{\log T}{\log x}\Big) + \; O\Big(\log(2x)\min\Big\{\frac{T}{x}, \frac{1}{\langle x\rangle}\Big\}\Big) \;+\; O\big(\log^4 T\ \big).$$

It would be interesting to have a version of Theorem 1.1 when $a = 1$ also. This looks possible but rather complicated and is not needed for the applications below.

Steuding [13] used (1.3) to prove the interesting result that the fractional parts of the sequence $\{\lambda\gamma_a\}_{\gamma_a>0}$ are uniformly distributed modulo 1, where λ is any fixed nonzero real number[1]. Our uniform versions of (1.3) allow us to prove a discrepancy estimate for this sequence.

Theorem 1.3. *Let $a \neq 0$ and let $\lambda \neq 0$ be a fixed real number. Then for T sufficiently large we have*

$$\sup_{0\leq\alpha\leq 1}\left|\frac{1}{N_a(T)}\left(\sum_{\substack{0<\gamma_a\leq T,\ \beta_a>0\\ \{\lambda\gamma_a\}\leq\alpha}} 1\right) \;-\; \alpha\right| \ll \frac{1}{\log\log T}, \tag{1.6}$$

where $\{x\}$ denotes the fractional part of the real number x.

The analogous problem for the zeros (i.e. the "case $a = 0$" of Theorem 1.3) has been studied extensively. The interested reader is referred to the survey [13] and the references therein for an informative discussion of this problem and related results.

As another application of Theorem 1.2 we prove

Theorem 1.4. *Let*

$$A(s) = \sum_{n\leq N} a(n)n^{-s},$$

[1]The statement of Theorem 6 in [13] is incorrect when $a = 1$ because in that case $-\zeta'(s)/(\zeta(s) - 1)$ cannot be expressed as an ordinary Dirichlet series.

where the $a(n)$ are complex numbers such that $|a(n)| \ll n^{\varepsilon}$ and $N = T^{\theta}$ with $T \geq 2$ and $0 < \theta < 1$ fixed. Then for $a \neq 0$ we have

$$\sum_{\substack{0<\gamma_a\leq T\\ \beta_a>0}} A(\rho_a) \;=\; \frac{T}{2\pi}\left(a(1)\log T - \sum_{2\leq n\leq N} \frac{a(n)\Lambda(n)}{n}\right) +\; O(T).$$

Specializing the Dirichlet polynomial $A(s)$ leads to the following formulae.

Corollary 1.1. *Let*

$$M(s) \;=\; \sum_{n\leq N} \frac{\mu(n)}{n^s} \qquad \textit{and} \qquad P(s) \;=\; \sum_{n\leq N} \frac{1}{n^s}.$$

If $N = T^{\theta}$ with $0 < \theta < 1$ fixed and $a \neq 0$, then

$$\sum_{\substack{0<\gamma_a\leq T\\ \beta_a>0}} M(\rho_a) \;=\; (1+\theta)\frac{T}{2\pi}\log T \;+\; O(T) \tag{1.7}$$

and

$$\sum_{\substack{0<\gamma_a\leq T\\ \beta_a>0}} P(\rho_a) \;=\; (1-\theta)\frac{T}{2\pi}\log T \;+\; O(T). \tag{1.8}$$

These results seemed counterintuitive to us at first. To the extent that one expects $M(s)$ to approximate $1/\zeta(s)$ and $P(s)$ to approximate $\zeta(s)$ on average, one might expect the first sum to be large and the second small when $|a|$ is small, and expect the reverse to be true when $|a|$ is large. However, from the corollary we see that the first sum is always larger than the second. The explanation seems to be that many a-points are quite close to zeros of $\zeta(s)$. In fact, the same argument as in the proof of the corollary shows that (1.7) and (1.8) hold with the ρ_a's replaced by ρ's.

Theorems 1.1 and 1.2 are proved by calculating the integrals

$$\frac{1}{2\pi i}\int_{\mathscr{R}} x^{\pm s}\frac{\zeta'(s)}{\zeta(s)-a}ds$$

over an appropriate rectangle $\mathscr{R}$. The size of the coefficients of the Dirichlet series for $1/(\zeta(s)-a)$, and its abscissae of convergence and absolute convergence enter into this analysis, so we shall also prove the following results. Although we do not require as much detail as the next two theorems provide, we record them in the hope that they may prove useful to others.

Theorem 1.5. *For $a \neq 0, 1$ the coefficients of the Dirichlet series*

$$\frac{1}{\zeta(s)-a} = \sum_{n=1}^{\infty} \frac{b_a(n)}{n^s}$$

are given by

$$b_a(n) = \begin{cases} -\sum_{k=0}^{\infty} a^{-k-1} d_k(n) & \textit{if} \quad |a| > 1, \\ \;\;\sum_{k=1}^{\infty} a^{k-1} d_{-k}(n) & \textit{if} \quad 0 < |a| < 1, \\ -\sum_{k=0}^{\infty} (a-1)^{-k-1} e_k(n) & \textit{if} \quad |a| = 1, \;\textit{but}\; a \neq 1. \end{cases}$$

Here $d_l(n)$ is the nth Dirichlet coefficient of $\zeta(s)^l$ and $e_l(n)$ is the nth Dirichlet coefficient of $(\zeta(s)-1)^l$. When $a=1$ the series is the generalized Dirichlet series

$$\frac{1}{\zeta(s)-1}=\sum_{\substack{m=1\\ \text{odd}}}^{\infty}\sum_{r=-\infty}^{\infty}\frac{b_1(m2^r)}{(m2^r)^s}$$

with coefficients

$$b_1(m2^r)=\sum_{\substack{l-k-1=r\\ k,l\geq 0}}(-1)^k f_k(2^l m),$$

where $f_k(n)$ is the nth Dirichlet coefficient of $(\zeta(s)-1-2^{-s})^k$.

Theorem 1.6. *Let $a\neq 0$. Define $\sigma^*>1$ to be the unique solution to the equation*

$$\begin{cases}\zeta(\sigma)=|a| & \text{if}\quad |a|>1,\\ \zeta(\sigma)=1+|1-a| & \text{if}\quad |a|=1, a\neq 1,\\ \dfrac{\zeta(2\sigma)}{\zeta(\sigma)}=|a| & \text{if}\quad |a|<1,\\ \zeta(\sigma)=1+2^{1-\sigma} & \text{if}\quad a=1.\end{cases}$$

Then the abscissa of absolute convergence $\overline{\sigma}$ of the series for $1/(\zeta(s)-a)$ satisfies

$$\overline{\sigma}\leq\sigma^*.$$

Remark. When $a=1$, $\sigma^*\approx 2.4241$.

Theorem 1.7. *Let $a\neq 0,1$ and let σ_0 be the abscissa of convergence of the Dirichlet series*

$$\frac{1}{\zeta(s)-a}=\sum_{n=1}^{\infty}\frac{b_a(n)}{n^s}.$$

Then $\sigma_0=\beta_a^*$, with β_a^* as in (1.5), and

$$1<\sigma_0\leq\overline{\sigma}\leq\sigma^*.$$

Moreover, for every $\varepsilon>0$

$$b_a(n)\ll n^{\beta_a^*+\varepsilon}.$$

This bound is sharp in the sense that

$$|b_a(n)|>n^{\beta_a^*-\varepsilon}$$

for infinitely many n.

2. Proof of Theorem 1.1

In the following proof we shall appeal to Theorem 1.7 though it is proved later.

The functional equation for $\zeta(s)$ is

$$\zeta(s)=\chi(s)\zeta(1-s), \tag{2.1}$$

where, by Stirling's formula,

$$\chi(s) \;=\; \Big(\frac{t}{2\pi}\Big)^{\frac{1}{2}-s}e^{i\pi/4+it}\Big(1+O\Big(\frac{1}{t}\Big)\Big) \tag{2.2}$$

as $t\to\infty$ in any fixed vertical strip. From (3.11.8) of [15], we have

$$|\zeta(s)| \;\gg\; \frac{1}{\log t}$$

as $t \to \infty$ when $\sigma \ge 1 - \frac{A}{\log t}$ and, in particular, when $\sigma \ge 1$. Thus, from (2.1) and (2.2) we have

$$|\zeta(s)| \gg \frac{t^{1/2-\sigma}}{\log t} \tag{2.3}$$

as $t \to \infty$ in any fixed vertical strip with $\sigma \le 0$. We may therefore choose a number $T_0 \ge 2$ such that $|\zeta(s)| > |a|$ for $\sigma \le 0$ and $t \ge T_0$, and also so that no γ_a equals T_0. With this T_0, and any $T > T_0$ with $T \ne \gamma_a$ for any γ_a, consider the contour integral

$$\begin{aligned} I &= \frac{1}{2\pi i}\bigg(\int_{B+1+iT_0}^{B+1+iT} + \int_{B+1+iT}^{-2+iT} + \int_{-2+iT}^{-2+iT_0} + \int_{-2+iT_0}^{B+1+iT_0}\bigg)\Big(\frac{\zeta'(s)}{\zeta(s)-a}\Big)x^s\,ds \\ &= I_1 + I_2 + I_3 + I_4, \end{aligned}$$

say. By the calculus of residues

$$I = \sum_{\substack{T_0<\gamma_a<T\\ \beta_a>0}} x^{\rho_a}.$$

To prove the Theorem we estimate I_1 through I_4.

To estimate I_1 we use the Dirichlet series expansion (1.4), which by Theorem 1.7 is absolutely convergent for $\sigma = B+1$, and integrate term-by-term. This leads to

$$\begin{aligned} I_1 &= -\sum_{n=2}^{\infty}\Lambda_a(n)\left(\frac{x}{n}\right)^{B+1}\left(\frac{1}{2\pi}\int_{T_0}^{T}\left(\frac{x}{n}\right)^{it}dt\right) \\ &= -\frac{T-T_0}{2\pi}\Lambda_a(x) + O\Bigg(\sum_{\substack{n=2\\ n\ne x}}^{\infty}\frac{|\Lambda_a(n)|}{n^{B+1}}x^{B+1}\min\Big\{T, \frac{1}{|\log(x/n)|}\Big\}\Bigg). \end{aligned} \tag{2.4}$$

To estimate the sum in the error term note that $|\log(x/n)| \gg 1$ for $n \le x/2$ or $n \ge 2x$. Thus the part of the sum with $n \le\ x/2$ or $n \ge 2x$ is

$$\ll \sum_{n=1}^{\infty}\frac{|\Lambda_a(n)|}{n^{B+1}}x^{B+1} \ll x^{B+1}. \tag{2.5}$$

The part with $x/2 < n < x$ is

$$\begin{aligned} &\ll \sum_{\frac{x}{2}<n<x}|\Lambda_a(n)|\min\Big\{T, \frac{1}{\log(x/n)}\Big\} \\ &= \sum_{\frac{x}{2}<n<N}|\Lambda_a(n)|\min\Big\{T, \frac{1}{\log(x/n)}\Big\} + |\Lambda_a(N)|\min\Big\{T, \frac{1}{\log(x/N)}\Big\}, \end{aligned}$$

where N is the largest integer less than x. By Theorem 1.7, we have $\Lambda_a(n) \ll_\varepsilon n^B$. Thus, since

$$\log\frac{x}{n} = -\log\Big(1-\frac{x-n}{x}\Big) > \frac{N-n}{x},$$

we see that

$$\sum_{\frac{x}{2}<n<N}|\Lambda_a(n)|\min\Big\{T, \frac{1}{\log x/n}\Big\} \le x\sum_{\frac{x}{2}<n<N}\frac{|\Lambda_a(n)|}{N-n} \ll_\varepsilon xN^B\log x \ll_\varepsilon x^{B+1}.$$

On the other hand, we have

$$|\Lambda_a(N)|\min\Big\{T, \frac{1}{\log x/N}\Big\} \ll_\varepsilon N^B\min\Big\{T, \frac{x}{x-N}\Big\} \ll x^B\min\Big\{T, \frac{x}{\langle x\rangle}\Big\}.$$

Hence the part with $x/2 < n < x$ is

$$\ll_\varepsilon x^{B+1} + x^B\min\Big\{T, \frac{x}{\langle x\rangle}\Big\}.$$

A similar argument gives the same estimate for the part with $x < n < 2x$. Using this and (2.5) in (2.4), we obtain

$$I_1 \;=\; -\frac{T}{2\pi}\Lambda_a(x) \;+\; O_\varepsilon\Big(x^{B+1} \;+\; x^B \min\Big\{T, \frac{x}{\langle x\rangle}\Big\}\Big). \tag{2.6}$$

To estimate I_2, we require the following lemma.

Lemma 2.1. *There is a positive number R_a depending only on a such that for $R \ge R_a$ we have*

$$\frac{\zeta'(s)}{\zeta(s)-a} \;=\; \sum_{|\rho_a - s|<R} \frac{1}{s-\rho_a} \;+\; O_R(\log t)$$

uniformly for $-2 \le \sigma \le R-2$ and large t.

Proof. Let $f(s) = \zeta(s) - a$. If $r_a > \beta_a^*$ is large enough, then for $\sigma_0 \ge r_a$ we will have $|f(\sigma_0 + it)| \gg_{\sigma_0} 1$ for all large t. We will show how to determine such an r_a later. We apply Lemma α of §3.9 in [15] with $f(s) = \zeta(s) - a$, $s_0 = \sigma_0 + iT$, $r = 4(\sigma_0 + 2)$, and T large. By the Phragmen-Lindelöf theorem applied to $\zeta(s)$ (see, for example, Chapter 5 of [14]), we have $f(s) = O_r(T^A)$ for some constant A uniformly for $|s - s_0| \le r$. Thus

$$\left|\frac{f(s)}{f(s_0)}\right| \;\ll_r\; T^A$$

uniformly for $|s - s_0| \le r$. It now follows from Lemma α that

$$\frac{\zeta'(s)}{\zeta(s)-a} \;=\; \sum_{|\rho_a - s|\le r/4} \frac{1}{s-\rho_a} \;+\; O_r(\log T)$$

for $|s - s_0| \le r/4$. If $s = \sigma + iT$ and $-2 \le \sigma \le 2\sigma_0 + 2$, then $|s - s_0| \le r/4$ because $r = 4(\sigma_0 + 2)$. This proves the lemma with $R_a = 4(r_a + 2)$ and $R = r/4$.

We now show how to choose an r_a such that if $\sigma_0 \ge r_a$ then $|f(\sigma_0 + it)| \gg_{\sigma_0} 1$ for all large t. If $a \ne 1$, then $|1 - a| \ne 0$. Hence, since $\lim_{\sigma\to 1}\zeta(\sigma) = 1$, we may choose a number σ_1 so large that $|1 - a| > \zeta(\sigma) - 1$ for $\sigma \ge \sigma_1$. If $a = 1$, we choose $\sigma_1 = 4$. In that case $\sigma \ge \sigma_1$ implies

$$\sum_{n=3}^{\infty} \frac{1}{n^\sigma} \;\le\; \int_2^\infty \frac{1}{u^\sigma}\, d\sigma \;=\; \frac{2^{1-\sigma}}{\sigma - 1} \;\le\; \frac{2}{3}\cdot\frac{1}{2^\sigma},$$

which in turn implies that

$$|\zeta(s) - 1| = \Big|2^{-s} + \sum_{n=3}^{\infty} n^{-s}\Big| \ge 2^{-\sigma}/3.$$

We now set $r_a = \max\{\sigma_1, \beta_a^* + 1\}$. It then follows that if $a \ne 1$ and $\sigma_0 \ge r_a$, then

$$|f(\sigma_0 + it)| = |\zeta(\sigma_0 + it) - 1 + (1-a)| \ge |1-a| - \zeta(\sigma_0) + 1 > 0.$$

On the other hand, if $a = 1$ and $\sigma_0 \ge r_a$, then

$$|f(\sigma_0 + it)| = |\zeta(\sigma_0 + it) - 1| \ge 3^{-1}2^{-\sigma_0} > 0.$$

This completes the proof. □

By Lemma 2.1,

$$I_2 \;=\; \sum_{|\rho_a - s|<R_a} \frac{1}{2\pi i}\int_{B+1+iT}^{-2+iT} \frac{x^s}{s-\rho_a}\, ds \;+\; O\Big(\log T\int_{-2}^{B+1} x^\sigma\, d\sigma\Big).$$

The error term is

$$\ll\ \log T\, \frac{x^{B+1}}{\log x}.$$

To estimate the sum, note that by Cauchy's integral theorem we may replace the line segment of integration in each term by the semicircle above or below the segment depending on whether ρ_a lies below or above that segment. Thus, the sum is

$$\ll \sum_{|\rho_a - s|<R} x^{B+1}.$$

By (1.1) the number of terms in the sum is $O(\log T)$. Thus,

$$I_2 \ \ll\ x^{B+1}\log T\Big(1+\frac{1}{\log x}\Big). \tag{2.7}$$

To estimate I_3 note that by our choice of T_0, if $\sigma=-2$ and $t\ge T_0$, then

$$\frac{1}{\zeta(s)-a} \ =\ \frac{1}{\zeta(s)}\Big(\frac{1}{1-a/\zeta(s)}\Big) \ =\ \frac{1}{\zeta(s)}\sum_{k=0}^{\infty}\Big(\frac{a}{\zeta(s)}\Big)^k.$$

Thus

$$I_3 \ =\ \frac{1}{2\pi i}\int_{-2+iT}^{-2+iT_0} x^s\frac{\zeta'(s)}{\zeta(s)}\,ds \ +\ \frac{1}{2\pi i}\int_{-2+iT}^{-2+iT_0} x^s\frac{\zeta'(s)}{\zeta(s)}\sum_{k=1}^{\infty}\Big(\frac{a}{\zeta(s)}\Big)^k ds. \tag{2.8}$$

From the logarithmic derivative of the functional equation for $\zeta(s)$ (for example, see [7] or [4], pp.73, 80, 81]) we have

$$-\frac{\zeta'(s)}{\zeta(s)} \ =\ \frac{\zeta'(1-s)}{\zeta(1-s)} \ +\ \log\frac{t}{2\pi} \ +\ O\Big(\frac{1}{t}\Big) \tag{2.9}$$

in any half-strip $A_1\le\sigma\le A_2, t\ge 1$ that does not contain zeros of $\zeta(s)$. Thus, the first integral on the right-hand side of (2.8) equals

$$\frac{x^{-2}}{2\pi}\int_{T_0}^{T} x^{it}\frac{\zeta'}{\zeta}(3-it)\,dt \ +\ \frac{x^{-2}}{2\pi}\int_{T_0}^{T} x^{it}\log\frac{t}{2\pi}\,dt \ +\ O(x^{-2}\log T).$$

We insert the Dirichlet series for ζ'/ζ into the first integral here and integrate term-by-term, and in the second we integrate by parts. In this way we find that

$$\frac{1}{2\pi i}\int_{-2+iT}^{-2+iT_0} x^s\frac{\zeta'(s)}{\zeta(s)}\,ds \ \ll\ \frac{\log T}{x^2}+\min\Big\{\frac{T\log T}{x^2},\frac{\log T}{x^2\log x}\Big\}. \tag{2.10}$$

To estimate the second integral on the right-hand side of (2.8), note that by(2.3) we have

$$\zeta(-2+it)\ \gg\ \frac{t^{5/2}}{\log t}$$

for $t\ge T_0$. Also, by (2.9), we have

$$\frac{\zeta'(-2+it)}{\zeta(-2+it)}\ \ll\ \log t$$

for $t\ge T_0$. Hence

$$\frac{1}{2\pi i}\int_{-2+iT}^{-2+iT_0} x^s\frac{\zeta'(s)}{\zeta(s)}\sum_{k=1}^{\infty}\Big(\frac{a}{\zeta(s)}\Big)^k ds \ \ll\ x^{-2}\int_{T_0}^{T}\frac{\log^2 t}{t^{5/2}}\,dt \ \ll\ x^{-2}.$$

From this and (2.10) we obtain

$$I_3 \ \ll\ \frac{\log T}{x^2}+\min\Big\{\frac{T\log T}{x^2},\frac{\log T}{x^2\log x}\Big\}. \tag{2.11}$$

Finally, $\zeta'(s)/(\zeta(s)-a)$ is bounded on $[-2+iT_0, B+1+iT_0]$, so

$$I_4 \ll x^{B+1}.$$

Combining this, (2.6), (2.7), and (2.11), we find that for $T \geq T_0$

$$\sum_{\substack{T_0<\gamma_a<T \\ \beta_a>0}} x^{\rho_a} = -\frac{T}{2\pi}\Lambda_a(x) + O_\varepsilon\Big(x^{B+1} + x^B \min\Big\{T, \frac{x}{\langle x\rangle}\Big\}\Big) \tag{2.12}$$
$$+ O\Big(x^{B+1}\log T\Big(1+\frac{1}{\log x}\Big)\Big) + O\Big(\frac{\log T}{x^2} + \frac{\log T}{x^2}\min\Big\{T, \frac{1}{\log x}\Big\}\Big).$$

Recall that we have assumed $T \neq \gamma_a$ for any γ_a. To remove this assumption, observe that by (1.1), changing T by a bounded amount in (2.12) changes the value of the sum on the left-hand side by at most $O(x^B \log T)$. This is clearly no more than the resulting change on the right-hand side.

As we mentioned in the first paragraph of Section 1, all the nontrivial a-points lie in a strip of the form $0 < \sigma < A$. There are at most a finite number of these with $0 < \gamma_a \leq T_0$, hence

$$\sum_{\substack{0<\gamma_a\leq Y \\ \beta_a>0}} x^{\rho_a} \ll x^B \tag{2.13}$$

uniformly for $1 < Y \leq T_0$. Taking $Y = T_0$ and combining this with (2.12), we see that we may extend the sum on the left-hand side of (2.12) to run over all ρ_a with $0 < \gamma_a \leq T$ and $\beta_a > 0$. The resulting formula holds for $T \geq T_0 \geq 2$. To see that it also holds when T is between 1 and T_0, note that (2.13) holds with $Y = T$, and it right-hand side is bounded by the second error term on the right-hand side of (2.12). This completes the proof of the Theorem 1.1.

3. Proof of Theorem 1.2

Let $a \neq 0$ and suppose that $x > 1$. As in the proof of Theorem 1.1 (see below (2.3)), we can choose a $T_0 \geq 2$ such that $|\zeta(s)| > |a|$ for $\sigma \leq 0, t \geq T_0$, and such that no γ_a equals T_0. We also choose a $T > T_0$ which is not equal to any γ_a. With σ^* as in Theorem 1.6, we see by the calculus of residues that

$$\sum_{\substack{T_0<\gamma_a<T \\ \beta_a>0}} x^{-\rho_a} = \frac{1}{2\pi i}\left(\int_{\sigma^*+1+iT_0}^{\sigma^*+1+iT} + \int_{\sigma^*+1+iT}^{-\frac{1}{\log(3x)}+iT} + \int_{-\frac{1}{\log(3x)}+iT}^{-\frac{1}{\log(3x)}+iT_0} + \int_{-\frac{1}{\log(3x)}+iT_0}^{\sigma^*+1+iT_0}\right)\Big(\frac{\zeta'(s)}{\zeta(s)-a}\Big)x^{-s}\,ds$$
$$= I_1 + I_2 + I_3 + I_4,$$

say.

To estimate I_1 we first assume $a \neq 1$. Using the Dirichlet series expansion (1.4) and integrating term-by-term, we obtain

$$I_1 = -\sum_{n=2}^{\infty}\Lambda_a(n)\Big(\frac{1}{nx}\Big)^{\sigma^*+1}\Big(\frac{1}{2\pi}\int_{T_0}^{T}\Big(\frac{1}{nx}\Big)^{it}\,dt\Big)$$
$$\ll x^{-\sigma^*-1}\sum_{n=2}^{\infty}\frac{|\Lambda_a(n)|}{n^{\sigma^*+1}\log(nx)} < \frac{1}{x^{\sigma^*+1}\log x}\sum_{n=2}^{\infty}\frac{|\Lambda_a(n)|}{n^{\sigma^*+1}} \ll \frac{1}{x^{\sigma^*+1}\log x}.$$

Now assume $a = 1$. By (6.4) below we see that

$$I_1 = \int_{\sigma^*+1+iT_0}^{\sigma^*+1+iT}\frac{\zeta'(s)}{\zeta(s)-1}x^{-s}\,ds$$
$$= -\int_{\sigma^*+1+iT_0}^{\sigma^*+1+iT}\sum_{\nu=2}^{\infty}\frac{\log\nu}{\nu^s}\sum_{k=0}^{\infty}(-1)^k 2^{(k+1)s}\sum_{n=3^k}^{\infty}\frac{f_k(n)}{n^s}x^{-s}\,ds.$$

Note that by Theorem 1.6 the double series over k and n converges absolutely when $\sigma = \sigma^* + 1$. Hence

$$\begin{aligned} I_1 &= -\sum_{\nu=2}^{\infty}\sum_{k=0}^{\infty}\sum_{n=3^k}^{\infty} (\log\nu)(-1)^k f_k(n) \int_{\sigma^*+1+iT_0}^{\sigma^*+1+iT} \left(\frac{2^{k+1}}{xn\nu}\right)^s ds \\ &\ll \sum_{\nu=2}^{\infty}\sum_{k=0}^{\infty}\sum_{n=3^k}^{\infty} \frac{(\log\nu) f_k(n)}{\log\left(\frac{xn\nu}{2^{k+1}}\right)} \left(\frac{2^{k+1}}{xn\nu}\right)^{\sigma^*+1}. \end{aligned}$$

This is absolutely convergent because

$$\log\left(\frac{xn\nu}{2^{k+1}}\right) \geq \log\left(\frac{3^k\cdot 2}{2^{k+1}}\right) = k\log\frac{3}{2}$$

for $k \geq 1$, while

$$\log\left(\frac{xn\nu}{2}\right) \geq \log\left(\frac{xn\cdot 2}{2}\right) = \log xn \geq \log x > 0$$

for $k = 0$. Thus

$$I_1 \ll \frac{1}{x^{\sigma^*+1}\log x}, \tag{3.1}$$

which is the same as our estimate when $a \neq 1$.

To estimate I_2 we use Lemma 2.1 to write

$$I_2 = \sum_{|\rho_a - s|<R} \frac{1}{2\pi i}\int_{\sigma^*+1+iT}^{-\frac{1}{\log(3x)}+iT} \frac{x^{-s}}{s-\rho_a}\,ds + O\left(\log T\int_{-\frac{1}{\log(3x)}}^{\sigma^*+1} x^{-\sigma}\,d\sigma\right).$$

The error term is

$$\ll \log T\,\frac{x^{\frac{1}{\log(3x)}}}{\log x} \ll \frac{\log T}{\log x}.$$

To bound the sum, note that by Cauchy's integral theorem we may replace the path of integration in each term by the semicircle above or below the path depending on whether ρ_a lies below or above it. In this way we see that the sum is

$$\ll \sum_{|\rho_a - s|<R} x^{\frac{1}{\log(3x)}} \ll \sum_{|\rho_a - s|<R} 1 \ll \log T$$

by (1.1). Thus

$$I_2 \ll \log T\left(1 + \frac{1}{\log x}\right). \tag{3.2}$$

Next we come to I_3. Since $|\zeta(s)| > |a|$ when $\sigma \leq 0$ and $t \geq T_0$, we have

$$\frac{1}{\zeta(s)-a} = \frac{1}{\zeta(s)}\left(\frac{1}{1-a/\zeta(s)}\right) = \frac{1}{\zeta(s)}\sum_{k=0}^{\infty}\left(\frac{a}{\zeta(s)}\right)^k.$$

Hence

$$\begin{aligned} I_3 &= \frac{1}{2\pi i}\int_{-\frac{1}{\log(3x)}+iT}^{-\frac{1}{\log(3x)}+iT_0} x^{-s}\frac{\zeta'(s)}{\zeta(s)}\,ds + \frac{1}{2\pi i}\int_{-\frac{1}{\log(3x)}+iT}^{-\frac{1}{\log(3x)}+iT_0} x^{-s}\frac{\zeta'(s)}{\zeta(s)}\sum_{k=1}^{\infty}\left(\frac{a}{\zeta(s)}\right)^k ds \\ &= I_{31} + I_{32}, \end{aligned}$$

say.

We first consider I_{32}. By (3.11.7) of [15] and (2.9) we have

$$\frac{\zeta'(s)}{\zeta(s)} \ll \log t \tag{3.3}$$

for $\sigma \leq 0$ bounded and $t \geq T_0$. Using this and (2.3), we see that the terms in I_{32} with $k > 1$ contribute at most

$$\ll a^2 \int_{T_0}^{T} \frac{\log^3 t}{t^{1+2/\log(3x)}}\, dt \ll \log^4 T. \tag{3.4}$$

By integration by parts, the term with $k = 1$ is

$$-\frac{a}{2\pi i} \int_{-\frac{1}{\log(3x)}+iT_0}^{-\frac{1}{\log(3x)}+iT} x^{-s} \frac{\zeta'(s)}{\zeta^2(s)}\, ds \;= a\frac{x^{-s}}{2\pi i\, \zeta(s)}\Big|_{-\frac{1}{\log(3x)}+iT_0}^{-\frac{1}{\log(3x)}+iT} + a\frac{\log x}{2\pi i} \int_{-\frac{1}{\log(3x)}+iT_0}^{-\frac{1}{\log(3x)}+iT} \frac{x^{-s}}{\zeta(s)}\, ds.$$

By (2.3) the first term on the right-hand side is $\ll T_0^{-1/2} \log T_0 \ll 1$. Hence,

$$I_{3,2} = a\frac{\log x}{2\pi i} \int_{-\frac{1}{\log(3x)}+iT_0}^{-\frac{1}{\log(3x)}+iT} \frac{x^{-s}}{\zeta(s)}\, ds + O(\log^4 T). \tag{3.5}$$

Using the functional equation 2.1 in the integral and switching the order of summation and integration (by absolute convergence), we see that

$$\begin{aligned} I_{3,2} =& a\frac{\log x}{2\pi i} \int_{-\frac{1}{\log(3x)}+iT_0}^{-\frac{1}{\log(3x)}+iT} \frac{x^{-s}}{\chi(s)} \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{1-s}} \right) ds + O(\log^4 T) \\ =& a \log x \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left(\frac{1}{2\pi i} \int_{-\frac{1}{\log(3x)}+iT_0}^{-\frac{1}{\log(3x)}+iT} \left(\frac{x}{n}\right)^{-s} \frac{1}{\chi(s)}\, ds \right) + O(\log^4 T). \end{aligned}$$

By (2.2), we next obtain

$$\begin{aligned} I_{3,2} =& \frac{ae^{-i\pi/4}}{2\pi} x^{1/\log 3x} \log x \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{1+1/\log(3x)}} \left(\int_{T_0}^{T} \left(\frac{t}{2\pi}\right)^{-\frac{1}{2}-\frac{1}{\log 3x}} \exp\left(it \log \frac{tn}{2\pi ex}\right)\left(1 + O\Big(\frac{1}{t}\Big)\right) dt \right) \\ &+ O(\log^4 T). \end{aligned}$$

The O-term inside the integral contributes

$$\ll \log 3x \sum_{n=1}^{\infty} \frac{1}{n^{1+1/\log(3x)}} \left(\int_{T_0}^{T} t^{-\frac{3}{2}}\, dt \right) \ll T_0^{-\frac{1}{2}} \log^2(3x) \ll \log^2 T.$$

Thus

$$\begin{aligned} I_{3,2} =& \frac{ae^{-i\pi/4}}{2\pi} x^{1/\log 3x} \log x \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{1+1/\log(3x)}} \left(\int_{T_0}^{T} \left(\frac{t}{2\pi}\right)^{-\frac{1}{2}-\frac{1}{\log 3x}} \exp\left(it \log \frac{tn}{2\pi ex}\right) dt \right) \\ &+ O(\log^4 T). \end{aligned}$$

We next split the interval of integration into dyadic intervals $I_k = (T/2^{k+1}, T/2^k]$ with $k = 0, 1, 2, \ldots, K = [(\log(T/T_0)/\log 2)] - 1$, plus the possible additional interval $I_K + 1 = [T_0, T/2^{K+1}] \subseteq [T_0, 2T_0]$. We then have

$$\begin{aligned} I_{3,2} =& \frac{ae^{-i\pi/4}}{2\pi} x^{1/\log 3x} \log x \sum_{n=1}^{\infty} \frac{\mu(n)}{n^{1+1/\log(3x)}} \left(\sum_{k=0}^{K+1} \mathcal{I}_k(n) \right) + O(\log^4 T) \\ =& \frac{ae^{-i\pi/4}}{2\pi} x^{1/\log 3x} \log x \sum_{k=0}^{K+1} \left(\sum_{n=1}^{\infty} \frac{\mu(n)}{n^{1+1/\log(3x)}} \mathcal{I}_k(n) \right) + O(\log^4 T), \end{aligned} \tag{3.6}$$

where

$$\mathcal{I}_k(n) = \int_{I_k} \left(\frac{t}{2\pi}\right)^{-\frac{1}{2}-\frac{1}{\log 3x}} \exp\left(it \log \frac{tn}{2\pi ex}\right) dt. \tag{3.7}$$

To estimate this we apply the following minor modification of a lemma in Gonek [5].

Lemma 3.1. *For large A and B with $A < r \le B \le 2A$,*

$$\int_A^B \exp\Big(it\log\Big(\frac{t}{re}\Big)\Big)\Big(\frac{t}{2\pi}\Big)^{a-\frac12}\,dt = (2\pi)^{1-a}r^a e^{-ir+\pi i/4} + O(E(r,A,B)),$$

where a is bounded and where

$$E(r,A,B) = A^{a-\frac12} + \frac{A^{a+\frac12}}{|A-r|+A^{\frac12}} + \frac{B^{a+\frac12}}{|B-r|+B^{\frac12}}. \tag{3.8}$$

For $r \le A$ or $r > B$,

$$\int_A^B \exp\Big(it\log\Big(\frac{t}{re}\Big)\Big)\Big(\frac{t}{2\pi}\Big)^{a-\frac12}\,dt = O(E(r,A,B)). \tag{3.9}$$

For us the cruder bound $E(r,A,B) \ll A^a$ suffices. Assuming that T_0 is sufficiently large (as we may) we then find that for $k = 0,\dots,K$,

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{1+1/\log(3x)}}\mathcal{I}_k(n) &\ll \sum_{\pi x2^{k+1}/T\le n<\pi x 2^{k+2}/T}\frac1n + \sum_{n=1}^{\infty}\frac{1}{n^{1+1/\log(3x)}}E\Big(\frac{2\pi x}{n},\frac{T}{2^{k+1}},\frac{T}{2^k}\Big)\\ &\ll \sum_{n\le x}\frac1n + \Big(\frac{T}{2^k}\Big)^{-\frac{1}{\log(3x)}}\sum_{n=1}^{\infty}\frac{1}{n^{1+1/\log(3x)}}\\ &\ll \log(3x)\Big(\Big(\frac{T}{2^k}\Big)^{-\frac{1}{\log(3x)}}+1\Big).\end{aligned} \tag{3.10}$$

We similarly find that

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^{1+1/\log(3x)}}\mathcal{I}_{K+1}(n) \ll \log(3x).$$

Inserting these estimates in (3.6) and summing, we find that

$$\begin{aligned}I_{3,2} &\ll \log^2(3x)\sum_{k=0}^{K+1}\Big(\Big(\frac{2^k}{T}\Big)^{\frac{1}{\log(3x)}}+1\Big) + \log^4 T \ll \log^3(3x)\Big(\Big(\frac{2^K}{T}\Big)^{\frac{1}{\log(3x)}}+K\Big)+\log^4 T\\ &\ll \log^3(3x)\Big(\Big(\frac{1}{T_0}\Big)^{\frac{1}{\log(3x)}}+\log T\Big)+\log^4 T \ll \log^4 T.\end{aligned}$$

To estimate I_{31}, we use (2.9) to write

$$\begin{aligned}I_{31} \;=\; &\frac{1}{2\pi i}\int_{-\frac{1}{\log(3x)}+iT_0}^{-\frac{1}{\log(3x)}+iT} x^{-s}\frac{\zeta'}{\zeta}(1-s)\,ds\\ &+\int_{-\frac{1}{\log(3x)}+iT_0}^{-\frac{1}{\log(3x)}+iT} x^{-s}\log\frac{t}{2\pi}\,ds \;+\; O\Big(\int_{T_0}^{T}\frac{dt}{t}\Big).\end{aligned}$$

Integrating by parts, we see that

$$\int_{T_0}^{T} x^{-it}\log\frac{t}{2\pi}\,dt \;\ll\; \frac{\log T}{\log x}.$$

Hence

$$I_{31} \;=\; \frac{1}{2\pi i}\int_{-\frac{1}{\log(3x)}+iT_0}^{-\frac{1}{\log(3x)}+iT} x^{-s}\frac{\zeta'}{\zeta}(1-s)\,ds \;+\; O\Big(\frac{\log T}{\log x}\Big)+O(\log T).$$

The remaining integral equals

$$-\frac{1}{x}\sum_{n=2}^{\infty}\Lambda(n)\Big(\frac{x}{n}\Big)^{1+\frac{1}{\log(3x)}}\left(\frac{1}{2\pi}\int_{T_0}^{T}\Big(\frac{x}{n}\Big)^{-it}\,dt\right)$$
$$= -\frac{T-T_0}{2\pi x}\Lambda(x) \;+\; O\Bigg(\frac{1}{x}\sum_{\substack{n=2\\ n\neq x}}^{\infty}\Lambda(n)\Big(\frac{x}{n}\Big)^{1+\frac{1}{\log(3x)}}\min\left\{T,\frac{1}{|\log(x/n)|}\right\}\Bigg).$$

By Lemma 2 of [7] this equals

$$-\frac{T-T_0}{2\pi x}\Lambda(x) \;+\; O(\log(2x)\log\log(3x)) \;+\; O\Big(\log(2x)\min\Big\{\frac{T}{x},\frac{1}{\langle x\rangle}\Big\}\Big).$$

Hence,

$$\begin{aligned} I_{3,1} \;=\; &-\frac{T}{2\pi x}\Lambda(x) \;+\; O(\log(2x)\log\log(3x)) + \;O\Big(\log(2x)\min\Big\{\frac{T}{x},\frac{1}{\langle x\rangle}\Big\}\Big)\\ &+\;O\Big(\frac{\log T}{\log x}\Big)+O(\log T). \end{aligned}$$

Combining our estimates for $I_{3,1}$ and $I_{3,2}$, we obtain

$$\begin{aligned} I_3 \;=\; &-\frac{T}{2\pi x}\Lambda(x) \;+\; O(\log(2x)\log\log(3x)) + \;O\Big(\log(2x)\min\Big\{\frac{T}{x},\frac{1}{\langle x\rangle}\Big\}\Big)\\ &+\;O\Big(\frac{\log T}{\log x}\Big)+O(\log^4 T). \end{aligned} \tag{3.11}$$

Finally, since $\zeta'(s)/(\zeta(s)-a)$ is bounded on $[-2+iT_0,\sigma^*+1+iT_0]$,

$$I_4 \;\ll\; x^{\frac{1}{\log(3x)}} \;\ll\; 1.$$

It follows from this, (3.1), (3.2), and (3.11) that

$$\sum_{\substack{T_0<\gamma_a<T\\ \beta_a>0}} x^{-\rho_a} \;=\; -\frac{T}{2\pi x}\Lambda(x) \;+\; O\Big(\frac{\log T}{\log x}\Big) + \;O\Big(\log(2x)\;\min\Big\{\frac{T}{x},\frac{1}{\langle x\rangle}\Big\}\Big)+O(\log^4 T). \tag{3.12}$$

To complete the proof of the theorem, we argue in much the same way as at the end of the proof of Theorem 1.1. That is, we first remove the constraint that no γ_a equals T and then note that we may extend the sum on the left-hand side of (3.12) to include the a-points with $0<\gamma_a\le T_0$. Finally, it is easy to see that we may replace our condition that $T>T_0$ by $T>1$.

4. The proof of Theorem 1.3

Levinson [10] has shown that for $\delta>0$ and T sufficiently large (depending on a), the number of a-points $\rho_a=\beta_a+i\gamma_a$ with $|\beta_a-\frac12|>\delta$ and $T\le\gamma_a\le 2T$ is $O(\delta^{-1}T\log\log T)$. Thus,

$$\begin{aligned}\sum_{T<\gamma_a\le 2T}|\beta_a-\tfrac12| \;&=\; \sum_{\substack{T<\gamma_a\le 2T\\ |\beta_a-\frac12|>\delta}}|\beta_a-\tfrac12| \;+\; \sum_{\substack{T<\gamma_a\le 2T\\ |\beta_a-\frac12|\le\delta}}|\beta_a-\tfrac12|\\ &\ll\; \frac{T\log\log T}{\delta} \;+\; \delta N_a(T).\end{aligned}$$

Taking $\delta=(\log\log T/\log T)^{1/2}$, we deduce that

$$\sum_{T<\gamma_a\le 2T}|\beta_a-\tfrac12| \;\ll\; T\sqrt{\log T\log\log T}. \tag{4.1}$$

Since $e^y-1\ll|y|\max\{1,e^y\}$ for any $y>0$, we see that

$$|x^{-1/2}-x^{-\beta_a}| \;=\; x^{-1/2}|1-x^{\frac12-\beta_a}| \;\ll\; |\beta_a-\tfrac12||\log x|\max\{x^{-1/2},x^{-\beta_a}\}.$$

By the remark after (2.3), there is a number T_0 such that if $\gamma_a \ge T_0$, then $\beta_a > 0$. We may obviously also assume that T_0 is so large that (4.1) holds for $T \ge T_0$. It follows that if $x > 1$, then for these ρ_a we have $x^{-\beta_a} < 1$. Hence, for $x > 1$

$$|x^{-1/2} - x^{-\beta_a}| \ \ll\ |\beta_a - \tfrac{1}{2}| \log x.$$

This and (4.1) imply that

$$\sum_{T<\gamma_a\le 2T} x^{-\frac{1}{2}-i\gamma_a} \ = \ \sum_{T<\gamma_a\le 2T} x^{-\rho_a} \ + \ O\big(T\log x\sqrt{\log T\log\log T}\big)$$

for $x > 1$. Replacing T by $\frac{T}{2}, \frac{T}{4}, \frac{T}{8}, \ldots$ and summing, we see that

$$\sum_{T_0<\gamma_a\le T} x^{-\frac{1}{2}-i\gamma_a} \ = \ \sum_{T_0<\gamma_a\le T} x^{-\rho_a} \ + \ O\big(T\log x\sqrt{\log T\log\log T}\big).$$

Now fix $0 < \theta < 1$ and assume that $1 < x \le T^\theta$. From this and Theorem 1.2 we find that

$$\begin{aligned}\sum_{T_0<\gamma_a\le T} x^{-i\gamma_a} \ &= \ -\frac{T}{2\pi\sqrt{x}}\Lambda(x) \ + \ O\Big(\sqrt{x}\frac{\log T}{\log x}\Big) + \ O\big(\sqrt{x}\log^4 T\ \big) \\ &\quad + \ O\Big(\sqrt{x}\log(2x)\min\Big\{\frac{T}{x}, \frac{1}{\langle x\rangle}\Big\}\Big) \ + \ O\big(\sqrt{x}T\log x\sqrt{\log T\log\log T}\big).\end{aligned} \tag{4.2}$$

By the Erdös-Turán inequality (see [11], Chapter 1, Corollary 1.1), if K is a positive integer, $\lambda \ne 0$ is a real number, and $[\alpha,\beta]$ is a subinterval of $[0,1]$, then

$$\Bigg|\sum_{\substack{T_0<\gamma_a\le T\\ \{\lambda\gamma_a\}\in[\alpha,\beta]}} 1 \ - \ (\beta-\alpha)(N_a(T)-N_a(T_0))\Bigg| \ \le \ \frac{N_a(T)}{K+1} \ + \ 3\sum_{k\le K}\frac{1}{k}\Bigg|\sum_{T_0<\gamma_a\le T} e(k\lambda\gamma_a)\Bigg|. \tag{4.3}$$

Without loss of generality we may assume that $\lambda > 0$. Taking $x = \exp(2\pi k\lambda)$ with k a positive integer in (4.2), and then taking the complex conjugates of both sides of the resulting equation, we find that

$$\frac{1}{k}\sum_{T_0<\gamma_a\le T} e(k\lambda\gamma_a) \ \ll_\lambda \ \frac{T}{e^{\pi k\lambda}} \ + \ e^{\pi k\lambda}T\sqrt{\log T\log\log T}.$$

Inserting this into (4.3) and evaluating, we obtain

$$\Bigg|\sum_{\substack{T_0<\gamma_a\le T\\ \{\lambda\gamma_a\}\in[\alpha,\beta]}} 1 \ - \ (\beta-\alpha)(N_a(T)-N_a(T_0))\Bigg| \ \ll \ \frac{N_a(T)}{K} \ + \ TK \ + \ e^{\pi K\lambda}T\sqrt{\log T\log\log T}.$$

Note that including the terms (if any) with $0 < \gamma_a \le T_0$, $\beta_a > 0$, and $\{\lambda\gamma_a\} \in [\alpha,\beta]$ changes the left-hand side by at most $O(1)$. If we now choose $K = \big[\frac{1/2-\varepsilon}{\pi\lambda}(\log\log T)\big]$, we obtain

$$\Bigg|\frac{1}{N_a(T)}\sum_{\substack{0<\gamma_a\le T,\ \beta_a>0\\ \{\lambda\gamma_a\}\in[\alpha,\beta]}} 1 \ - \ (\beta-\alpha)\Bigg| \ \ll \ \frac{1}{\log\log T}$$

for $\lambda > 0$ fixed, and uniformly for any subinterval $[\alpha,\beta]$ of $[0,1]$. The estimate (1.6) follows easily from this.

5. Proof of Theorem 1.4 and Corollary 1.1

By (1.1) and Theorem 1.2 with $x = n$ an integer ≥ 2, we see that

$$\sum_{\substack{0<\gamma_a\le T\\ \beta_a>0}} \frac{1}{n^{\rho_a}} = -\frac{T}{2\pi n}\Lambda(n) + O(\log n) + O(\log^4 T).$$

Thus, since $N = T^\theta$ with $0 < \theta < 1$ fixed, we have

$$\begin{aligned}\sum_{\substack{0<\gamma_a\le T\\ \beta_a>0}} A(\rho_a) &= \sum_{n\le N} a(n) \sum_{\substack{0<\gamma_a\le T\\ \beta_a>0}} n^{-\rho_a}\\ &= a(1)N_a(T) + \sum_{2\le n\le N} a(n)\Big(-\frac{T}{2\pi n}\Lambda(n) + O(\log^4 T)\Big) \qquad (5.1)\\ &= \frac{T}{2\pi}\left(a(1)\log T - \sum_{2\le n\le N}\frac{a(n)\Lambda(n)}{n}\right) + \ O(T^{\theta+2\epsilon}).\end{aligned}$$

This gives Theorem 1.4, assuming ε is so small that $\theta + 2\varepsilon \le 1$.

To prove Corollary 1.1, first take $A(s) = M(s)$ in (5.1), where $M(s) = \sum_{n\le N}\mu(n)n^{-s}$ and $N = T^\theta$ with $0 < \theta < 1$ fixed. Then we find that

$$\sum_{\substack{0<\gamma_a\le T\\ \beta_a>0}} M(\rho_a) = \frac{T}{2\pi}\left(\log T - \sum_{2\le n\le N}\frac{\mu(n)\Lambda(n)}{n}\right) + \ O(T).$$

The sum over n equals

$$-\sum_{p\le N}\frac{\log p}{p} = \ -\log N + O(1).$$

Thus,

$$\sum_{\substack{0<\gamma_a\le T\\ \beta_a>0}} M(\rho_a) \ = \frac{T}{2\pi}\log T \ + \theta\frac{T}{2\pi}\log T \ + \ O(T),$$

which is the same as (1.7).

For $P(s) = \sum_{n\le N} n^{-s}$, we similarly find that

$$\begin{aligned}\sum_{\substack{0<\gamma_a\le T\\ \beta_a>0}} P(\rho_a) &= \frac{T}{2\pi}\left(\log T - \sum_{2\le n\le N}\frac{\Lambda(n)}{n}\right) + \ O(T)\\ &= \frac{T}{2\pi}(\log T - \log N) + O(T).\end{aligned}$$

This gives (1.8).

6. The proof of Theorems 1.5 and 1.6

As in the previous sections we assume $a \neq 0$ is a fixed complex number. Throughout this section we write

$$f(s) = \zeta(s) - a.$$

As we shall show, when $a \neq 1$ and σ is sufficiently large, $1/f(s)$ has a Dirichlet series representation

$$\frac{1}{f(s)} = \frac{1}{\zeta(s) - a} = \sum_{n=1}^{\infty}\frac{b_a(n)}{n^s}.$$

We shall also show that when $a = 1$ and σ is large, one has the generalized Dirichlet series representation

$$\frac{1}{f(s)} = \frac{1}{\zeta(s) - 1} = \sum_{\substack{m=1\\ \text{odd}}}^{\infty}\sum_{r=-\infty}^{\infty}\frac{b_1(m2^r)}{(m2^r)^s}. \tag{6.1}$$

We denote the abscissa of convergence of $1/f(s)$ by σ_0 and its abscissa of absolute convergence by $\overline{\sigma}$. Both, of course, depend on a and, in general, neither is easy to determine precisely. Theorem 1.5

gives explicit formulae for the coefficients $b_a(n)$ of $f(s)$ and Theorem 1.6 gives upper bounds for $\overline{\sigma}$. The two theorems are most conveniently proved together for the various ranges of a.

First we consider the case when $|a| > 1$. Clearly $\zeta(\sigma)$ decreases from ∞ to 1 as σ increases from 1 to ∞. Hence $\zeta(\sigma) = |a|$ has a unique solution $\sigma^* > 1$, and for $\sigma > \sigma^*$ we have $\zeta(\sigma) < \zeta(\sigma^*)$. Moreover, $|\zeta(s)| \le \zeta(\sigma)$ for $\sigma > 1$. Thus, when $\sigma > \sigma^*$

$$|\zeta(s)| \le \zeta(\sigma) < \zeta(\sigma^*) = |a|.$$

Furthermore, for $\sigma > \sigma^*$ we have

$$\frac{1}{\zeta(s) - a} = -\frac{1}{a} \sum_{k=0}^{\infty} \left(\frac{\zeta(s)}{a}\right)^k = -\frac{1}{a} \sum_{k=0}^{\infty} \frac{1}{a^k} \sum_{n=1}^{\infty} \frac{d_k(n)}{n^s}.$$

The double sum, in fact, converges absolutely since $d_k(n)$ is positive and

$$-\frac{1}{|a|} \sum_{k=0}^{\infty} \frac{1}{|a|^k} \sum_{n=1}^{\infty} \frac{d_k(n)}{n^\sigma} = \frac{1}{\zeta(\sigma) - |a|}.$$

Thus, when $|a| > 1$ we have $\overline{\sigma} \le \sigma^*$ and

$$b_a(n) = -\sum_{k=0}^{\infty} \frac{d_k(n)}{a^{k+1}}.$$

Remark. It is not difficult to see from the proof that when $a > 1$ is real, we in fact have $\overline{\sigma} = \sigma^*$.

Next we consider the case $0 < |a| < 1$. For $\sigma > 1$

$$|\zeta(s)| \ge \prod_p \left(1 + \frac{1}{p^\sigma}\right)^{-1} = \frac{\zeta(2\sigma)}{\zeta(\sigma)}.$$

Since $\zeta(2\sigma)/\zeta(\sigma)$ increases from 0 to 1 as σ increases from 1 to ∞, there is a unique solution $\sigma^* > 1$ of the equation $\zeta(2\sigma)/\zeta(\sigma) = |a|$, and if $\sigma > \sigma^*$, then $|\zeta(s)| \ge \zeta(2\sigma)/\zeta(\sigma) > |a|$. Thus, for $\sigma > \sigma^*$

$$\frac{1}{\zeta(s) - a} = \sum_{k=0}^{\infty} \frac{a^k}{\zeta(s)^{k+1}} = \sum_{k=0}^{\infty} a^k \sum_{n=1}^{\infty} \frac{d_{-(k+1)}(n)}{n^s}. \tag{6.2}$$

For any prime power p^j, we have $d_{-(k+1)}(p^j) = \binom{k+1}{j}(-1)^j$. Hence

$$\sum_{n=1}^{\infty} \frac{|d_{-(k+1)}(n)|}{n^\sigma} = \prod_p \left(\sum_{j=0}^{k+1} \binom{k+1}{j} \frac{1}{p^{j\sigma}}\right) = \left(\frac{\zeta(\sigma)}{\zeta(2\sigma)}\right)^{k+1}.$$

Therefore the double sum in (6.2) is absolutely convergent and has modulus

$$\le \sum_{k=0}^{\infty} |a|^k \left(\frac{\zeta(\sigma)}{\zeta(2\sigma)}\right)^{k+1} = \frac{\zeta(\sigma)}{\zeta(2\sigma)} \cdot \frac{1}{1 - |a|\zeta(\sigma)/\zeta(2\sigma)}.$$

It follows that $\overline{\sigma} \le \sigma^*$ and that

$$b_a(n) = \sum_{k=1}^{\infty} a^{k-1} d_{-k}(n).$$

Suppose next that $|a| = 1$ but $a \ne 1$. If $\sigma > 1$

$$|\zeta(s) - 1| \le \zeta(\sigma) - 1,$$

and the right-hand side decreases from ∞ to 0 as σ increases from 1 to ∞. Thus, there is a unique solution $\sigma^* > 1$ to the equation $\zeta(\sigma) - 1 = |a - 1|$. Moreover, if $\sigma > \sigma^*$, then $|\zeta(s) - 1| < \zeta(\sigma^*) - 1 = |a - 1|$. Hence, for $\sigma > \sigma^*$,

$$|\zeta(s) - 1| < |a - 1|.$$

We therefore see that

$$\frac{1}{\zeta(s)-a} = \frac{1}{(\zeta(s)-1)-(a-1)} = -\sum_{k=0}^{\infty} \frac{(\zeta(s)-1)^k}{(a-1)^{k+1}}$$
$$= -\sum_{k=0}^{\infty} \frac{1}{(a-1)^{k+1}} \sum_{n=1}^{\infty} \frac{e_k(n)}{n^s},$$

where

$$(\zeta(s)-1)^k = \sum_{n=1}^{\infty} \frac{e_k(n)}{n^s} \qquad (\sigma > 1). \tag{6.3}$$

We note that the $e_k(n) \geq 0$, so for $\sigma > \sigma^*$

$$\sum_{k=0}^{\infty} \frac{1}{|a-1|^{k+1}} \sum_{n=1}^{\infty} \frac{e_k(n)}{n^\sigma} = \sum_{k=0}^{\infty} \frac{(\zeta(\sigma)-1)^k}{|a-1|^{k+1}} = \frac{1}{|a-1|-(\zeta(\sigma)-1)}.$$

Thus $\overline{\sigma} \leq \sigma^*$, where σ^* is the unique solution to $\zeta(\sigma) = 1 + |1-a|$ in $\sigma > 1$. We also see that

$$b_a(n) = -\sum_{k=0}^{\infty} \frac{e_k(n)}{(a-1)^{k+1}},$$

where $e_k(n)$ is given by (6.3).

Finally, suppose that $a = 1$. Then for $\sigma > 1$

$$\frac{1}{\zeta(s)-1} = \frac{2^s}{1+(2/3)^s+(2/4)^s+\cdots}\ .$$

This time we let σ^* be the unique solution in $\sigma > 1$ of

$$1 = (2/3)^\sigma + (2/4)^\sigma + \cdots$$

or, equivalently, of

$$\zeta(\sigma) = 1 + 2^{1-\sigma}.$$

Then if $\sigma > \sigma^*$,

$$(2/3)^\sigma + (2/4)^\sigma + \cdots < 1$$

and we have

$$\begin{aligned}
\frac{1}{\zeta(s)-1} &= \frac{2^s}{1+(2/3)^s+(2/4)^s+\cdots} \\
&= \sum_{k=0}^{\infty} (-1)^k 2^{(k+1)s} \Big(\zeta(s) - 1 - \frac{1}{2^s}\Big)^k \qquad (6.4) \\
&= \sum_{k=0}^{\infty} (-1)^k 2^{(k+1)s} \sum_{n=3^k}^{\infty} \frac{f_k(n)}{n^s},
\end{aligned}$$

where

$$(\zeta(s)-1-2^{-s})^k = \sum_{n=3^k}^{\infty} \frac{f_k(n)}{n^s}.$$

By our choice of σ^*, the double series in (6.4) converges absolutely when $\sigma > \sigma^*$. Thus we have $\overline{\sigma} \leq \sigma^*$, and (6.1) holds with coefficients given by

$$b_1(m2^r) = \sum_{\substack{l-k-1=r \\ k,l \geq 0}} (-1)^k f_k(2^l m).$$

This completes our proof of Theorems 1.5 and 1.6.

7. The proof of Theorem 1.7

By a theorem of Landau [9] (Appendix, Satz 12), if

$$g(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$$

is convergent and nonzero for $\sigma > \alpha$ and $a_1 \neq 0$, then

$$\frac{1}{g(s)} = \sum_{n=1}^{\infty} \frac{b_n}{n^s}$$

converges for $\sigma > \alpha$. We apply this to the function $g(s) = \zeta(s) - a$ when $a \neq 1$ or 0. Let $\rho_a = \beta_a + i\gamma_a$ denote a typical zero of $\zeta(s) - a$ and let

$$\beta_a^* = \sup_{\rho_a} \beta_a,$$

as before. Then the series for $1/(\zeta(s) - a)$ converges when $\sigma > \beta_a^*$. In fact, β_a^* is the exact abscissa of convergence because $1/(\zeta(s) - a)$ has a pole at every zero ρ_a of $\zeta(s) - a$ and, therefore, the series cannot converge at ρ_a. Thus, we have $\sigma_0 = \beta_a^* \leq \overline{\sigma}$. Next recall that $\beta_a^* > 1$ (see just after (1.5)). From this and Theorem 1.6 we see that for $a \neq 0, 1$,

$$1 < \sigma_0 = \beta_a^* \leq \overline{\sigma} \leq \sigma^*.$$

Finally we turn to the growth of the coefficients $b_a(n)$. Since the terms $|b_a(n)/n^\sigma|$ must tend to zero when $\sigma > \beta_a^*$, it is clear that for any $\varepsilon > 0$

$$b_a(n) \ll n^{\beta_a^* + \varepsilon}.$$

By a theorem of Bombieri and Ghosh ([3], Theorem 3) this upper bound is sharp when $a \neq 0, 1$ in the sense that

$$|b_a(n)| > n^{\beta_a^* - \varepsilon}$$

for infinitely many n. this completes the proof of Theorem 1.7.

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